An Alternative Three-Parameter Mittag-Leffler Function without the Pochhammer Symbol

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Abstract: **The three-parameter Mittag-Leffler function introduced by Prabhakar commonly used today makes use of the Pochhammer symbol. In this paper, we introduce an alternative three-parameter Mittag-Leffler function that is without the Pochhammer symbol. It is shown that this new simplified version of the three-parameter Mittag-Leffler function behaves like the one commonly used in the literature. Also, we state some properties of this new three-parameter Mittag-Leffler function, so that it can be used in applications.**

Keywords: **Mitttag-Leffler function; the Pochhammer symbol.**

1. INTRODUCTION

The Mittag-Leffler function plays a very important role in the theory of fractional differential equations, just like the exponential function in the theory of integer-order differential equations. The Mittag-Leffler function is a generalization of the exponential function.

In 1903, the Swedish mathematician Gosta Mittag-Leffler [1] introduced the Mittag-Leffler function defined as:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, R(\alpha) > 0, z \in \mathbb{C} \qquad (1)$$

In 1905, Wiman [2] introduced the two-parameter Mittag-Leffler function defined as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, R(\alpha) > 0, R(\beta) > 0, z \in \mathbb{C} \qquad (2)$$

In 1971, the three-parameter Mittag-Leffler function, which is a generalization of (1) and (2) was introduced by Prabhakar [3] and defined as:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha, \beta, \gamma \in \mathbb{C}, R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0, z \in \mathbb{C}) \qquad (3)$$

where $(\gamma)_k$ is the Pochhammer symbol [4] defined by

$$(\gamma)_k = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)}$$

Several scholars [5-10] have introduced different generalizations of the multi-parameter Mittag-Leffler function.

2. THE MAIN RESULT

The alternative three-parameter Mittag-Leffler function without the Pochhammer symbol is defined as:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\gamma z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha, \beta, \gamma \in \mathbb{C}, R(\alpha) > 0, R(\beta) > 0, R(\gamma) > 0, z \in \mathbb{C}) \tag{4}$$

3. RELATION OF THE ALTERNATIVE FUNCTION WITH THE PRABHAKAR FUNCTION

In this section, we want to show how the new three-parameter Mittag-Leffler function we introduce in this work agrees with the Prabhakar function defined in (3), which is the commonly used three-parameter Mittag-Leffler function with the Pochhammer symbol.

For $\alpha = \beta = \gamma = 1$, (4) becomes

$$E_{1,1}^{1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{\mathrm{k}!} = e^z \tag{5}$$

Putting $\beta = \gamma = 1$ in (4), we obtain the one-parameter Mittag-Leffler function defined in (1):

$$E_{\alpha,1}^{1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} = E_\alpha(z) \tag{6}$$

Putting $\gamma = 1$ in (4), we obtain the two-parameter Mittag-Leffler function defined in (2):

$$E_{\alpha,\beta}^{1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = E_{\alpha,\beta}(z) \tag{7}$$

The results obtained in (5) – (7) using the alternative function (4) agree with those obtained using the commonly used Prabhakar function (3). This clearly shows that the new function proposed in this work is a fitting generalization exponential, one-parameter Mittag-Leffler and two-parameter Mittag-Leffler functions.

4. SOME PROPERTIES OF THE ALTERNATIVE MITTAG-LEFFLER FUNCTION

Theorem 1

Let $z \in \mathbb{C}$ and , $n \in \mathbb{N}$, then

$$E_{1,n}^{1}(z) = \frac{1}{z^{n-1}} \left(e^z - \sum_{k=0}^{n-2} \frac{z^k}{k!} \right) \tag{8}$$

Proof

From (4), we have that

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\gamma z^k}{\Gamma(\alpha k + \beta)}$$

Let $\alpha = \gamma = 1$ and $\beta = n$, then

$$E_{1,n}^{1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+n)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+n-1)!}$$

$$\Rightarrow E_{1,n}^{1}(z) = \frac{1}{z^{n-1}} \sum_{k=0}^{\infty} \frac{z^{k+n-1}}{(k+n-1)!} \tag{9}$$

$$\sum_{k=0}^{\infty} \frac{z^{k+n-1}}{(k+n-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} - \sum_{k=0}^{n-2} \frac{z^k}{k!} \tag{10}$$

Putting (10) into (9), we have:

$$E_{1,n}^{1}(z) = \frac{1}{z^{n-1}}\left(\sum_{k=0}^{\infty}\frac{z^k}{k!} - \sum_{k=0}^{n-2}\frac{z^k}{k!}\right)$$

$$E_{1,n}^{1}(z) = \frac{1}{z^{n-1}}\left(e^z - \sum_{k=0}^{n-2}\frac{z^k}{k!}\right)$$

Theorem 2

If $\alpha, \beta, \gamma \in \mathbb{C}, R(\alpha) > 0,\ R(\beta) > 0,\ R(\gamma) > 0, z \in \mathbb{C}$, and $r, n \in \mathbb{N}$, then

$$E_{\alpha,\beta}^{\gamma}(z) = \beta E_{\alpha,\beta+1}^{\gamma}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma}(z) \tag{11}$$

Proof

By definition (4),

$$\beta E_{\alpha,\beta+1}^{\gamma}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma}(z) = \beta\left(\sum_{k=0}^{\infty}\frac{\gamma z^k}{\Gamma(\alpha k+\beta+1)}\right) + \alpha z \frac{d}{dz}\left(\sum_{k=0}^{\infty}\frac{\gamma z^k}{\Gamma(\alpha k+\beta+1)}\right)$$

$$= \sum_{k=0}^{\infty}\frac{\beta\gamma z^k}{\Gamma(\alpha k+\beta+1)} + \sum_{k=0}^{\infty}\frac{\alpha k\gamma z^k}{\Gamma(\alpha k+\beta+1)}$$

$$= \sum_{k=0}^{\infty}\frac{\gamma z^k}{\Gamma(\alpha k+\beta+1)}(\alpha k+\beta)$$

$$= \sum_{k=0}^{\infty}\frac{\gamma z^k}{(\alpha k+\beta)\Gamma(\alpha k+\beta)}(\alpha k+\beta)$$

$$= \sum_{k=0}^{\infty}\frac{\gamma z^k}{\Gamma(\alpha k+\beta)}$$

$$= E_{\alpha,\beta}^{\gamma}(z)$$

Thus, $E_{\alpha,\beta}^{\gamma}(z) = \beta E_{\alpha,\beta+1}^{\gamma}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma}(z)$

Theorem 3

If $\alpha, \beta, \gamma \in \mathbb{C}, z \in \mathbb{C}$, then

$$E_{\alpha,\beta}^{\gamma}(z) = zE_{\alpha,\alpha+\beta}^{\gamma}(z) + \frac{\gamma}{\Gamma(\beta)} \tag{12}$$

Proof

By definition (4),

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty}\frac{\gamma z^k}{\Gamma(\alpha k+\beta)}$$

$$= \sum_{k=-1}^{\infty}\frac{\gamma z^{k+1}}{\Gamma(\alpha k+\alpha+\beta)}$$

$$= \frac{\gamma}{\Gamma(\beta)} + \sum_{k=0}^{\infty} \frac{\gamma z^k}{\Gamma(\alpha k + \alpha + \beta)}$$

$$= z \sum_{k=0}^{\infty} \frac{\gamma z^{k+1}}{\Gamma(\alpha k + \alpha + \beta)} + \frac{\gamma}{\Gamma(\beta)}$$

$$= z E^{\gamma}_{\alpha,\alpha+\beta}(z) + \frac{\gamma}{\Gamma(\beta)}$$

So, $E^{\gamma}_{\alpha,\beta}(z) = z E^{\gamma}_{\alpha,\alpha+\beta}(z) + \frac{\gamma}{\Gamma(\beta)}$

Theorem 4 (Derivative of the alternative Mittag-Leffler function)

If $\alpha, \beta, \gamma \in \mathbb{C}, R(\alpha) > 0,\ R(\beta) > 0,\ R(\gamma) > 0, z \in \mathbb{C}$, and $r, n \in \mathbb{N}$, then

$$\frac{d^n}{dz^n}\left[z^{\beta+r\alpha-1} E^{\gamma}_{\alpha,\beta+r\alpha}(z^\alpha)\right] = z^{\beta+r\alpha-n-1} E^{\gamma}_{\alpha,\beta+r\alpha-n}(z^\alpha) \tag{13}$$

Proof

By definition (4),

$$E^{\gamma}_{\alpha,\beta+r\alpha}(z^\alpha) = \sum_{k=0}^{\infty} \frac{\gamma z^{\alpha k}}{\Gamma(\alpha k + \beta + r\alpha)}$$

$$\Rightarrow z^{\beta+r\alpha-1} E^{\gamma}_{\alpha,\beta+r\alpha}(z^\alpha) = \sum_{k=0}^{\infty} \frac{\gamma z^{\alpha(k+r)+\beta-1}}{\Gamma(\alpha(k+r)+\beta)}$$

$$\frac{d^n}{dz^n}\left[z^{\beta+r\alpha-1} E^{\gamma}_{\alpha,\beta+r\alpha}(z^\alpha)\right] = \frac{d^n}{dz^n}\left[\sum_{k=0}^{\infty} \frac{\gamma z^{\alpha(k+r)+\beta-1}}{\Gamma(\alpha(k+r)+\beta)}\right]$$

$$= z^{\beta+r\alpha-n-1} \sum_{k=0}^{\infty} \frac{\gamma z^{\alpha k}}{\Gamma(\alpha k + \beta + r\alpha - n)}$$

$$= z^{\beta+r\alpha-n-1} E^{\gamma}_{\alpha,\beta+r\alpha-n}(z^\alpha)$$

Theorem 5 (Integration of the alternative Mittag-Leffler function)

If $\alpha, \beta, \gamma \in \mathbb{C}, R(\alpha) > 0,\ R(\beta) > 0,\ R(\gamma) > 0, z \in \mathbb{C}$, and $r, n \in \mathbb{N}$, then

$$\int_0^z z^{\beta+r\alpha-1} E^{\gamma}_{\alpha,\beta+r\alpha}(z^\alpha) dz = z^{\beta+r\alpha} E^{\gamma}_{\alpha,\beta+r\alpha+1}(z^\alpha) \tag{14}$$

Proof

By definition (4),

$$E^{\gamma}_{\alpha,\beta+r\alpha}(z^\alpha) = \sum_{k=0}^{\infty} \frac{\gamma z^{\alpha k}}{\Gamma(\alpha k + \beta + r\alpha)}$$

$$\Rightarrow z^{\beta+r\alpha-1} E^{\gamma}_{\alpha,\beta+r\alpha}(z^\alpha) = \sum_{k=0}^{\infty} \frac{\gamma z^{\alpha(k+r)+\beta-1}}{\Gamma(\alpha(k+r)+\beta)} \tag{15}$$

Integrating both sides of (15) w.r.t. z from 0 to z, we have:

$$\int_0^z z^{\beta+r\alpha-1} E^{\gamma}_{\alpha,\beta+r\alpha}(z^\alpha) dz = z^{\beta+r\alpha} \sum_{k=0}^{\infty} \frac{\gamma z^{\alpha k}}{\Gamma(\alpha k + \beta + r\alpha + 1)}$$

$$= z^{\beta+r\alpha} E^{\gamma}_{\alpha,\beta+r\alpha+1}(z^\alpha)$$

5. CONCLUSION

In this paper, we introduce an alternative three-parameter Mittag-Leffler function without using the Pochhammer symbol. The importance of this alternative function is that it satisfies the properties of the original function and provides new relations. Also, this alternative function is simpler in nature than one commonly used. Thus, we conclude this alternative three-parameter Mittag-Leffler function will find easy adaption in applications.

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